Readout Air Diffusion Noise more precise

Daniel Grass
(Dated: April 17, 2024)

1. INTEGRATE OVER LENGTHS

I am calculating the Readout Air diffusion noise $S_{\mathrm{L}}^{\mathrm{RAD}}(\Omega)$ but treating more precisely by considering the evolution of the beam radius. Previously, the equation has the following relation

$$S^{\mathrm{RAD_L}}(\Omega) \propto \frac{L}{w^4} \tag{1}$$

where L is the path length and w is the waist size (the 2 sigma radius).

To get the more accurate factor, integrate along the path from L_1 to L_2. This L/w^4 factor becomes

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} \tag{2}$$

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \tag{3}$$

where z_R is the Rayleigh range

$$z_R = \frac{\pi w_0^2 n}{\lambda}, \tag{4}$$

where n is the index of refraction and λ is the wavelength.

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \int_{L_1}^{L_2} \frac{dz}{w_0^4\left(1+\left(\frac{z}{z_R}\right)^2\right)^2} \tag{5}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{1}{w_0^4}\int_{L_1}^{L_2} \left(1+\left(\frac{z}{z_R}\right)^2\right)^{-2} dz \tag{6}$$

Let's do a u sub:

$$\tilde{z} \equiv \frac{z}{z_R} \tag{7}$$

$$d\tilde{z} = \frac{dz}{z_R} \tag{8}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{1}{w_0^4}\int_{L_1/z_R}^{L_2/z_R} \left(1+\tilde{z}^2\right)^{-2} z_R d\tilde{z} \tag{9}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4}\int_{L_1/z_R}^{L_2/z_R} \left(1+\tilde{z}^2\right)^{-2} d\tilde{z} \tag{10}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4}\left[\frac{tan^{-1}(\tilde{z})}{2} + \frac{\tilde{z}}{2\tilde{z}^2+2}\right]_{L_1/z_R}^{L_2/z_R} \tag{11}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4}\left[\frac{\tan^{-1}(\tilde{z})}{2} + \frac{\tilde{z}}{2\tilde{z}^2+2}\right]_{L_1/z_R}^{L_2/z_R} \tag{12}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{2w_0^4}\left(\tan^{-1}\left(\frac{L_2}{z_R}\right) + \frac{\left(\frac{L_2}{z_R}\right)}{\left(\frac{L_2}{z_R}\right)^2+1} - \tan^{-1}\left(\frac{L_1}{z_R}\right) - \frac{\left(\frac{L_1}{z_R}\right)}{\left(\frac{L_1}{z_R}\right)^2+1}\right) \tag{13}$$

This works, although it's pretty messy

2. INTEGRATE OVER GOUY PHASE

Can also evaluate (6) with Gouy phase, either here or earlier.

$$\varphi = \tan^{-1}\left(\frac{z}{z_R}\right) \tag{14}$$

Let's do a different u sub with φ:

$$\tan(\varphi) = \frac{z}{z_R} \tag{15}$$

$$\sec^2(\varphi)d\varphi = \frac{dz}{z_R} \tag{16}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{1}{w_0^4}\int_{\varphi_1}^{\varphi_2} \left(1+\tan^2(\varphi)\right)^{-2} z_R \sec^2(\varphi)d\varphi \tag{17}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4}\int_{\varphi_1}^{\varphi_2} \sec^{-4}(\varphi)\sec^2(\varphi)d\varphi \tag{18}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4}\int_{\varphi_1}^{\varphi_2} \cos^2(\varphi)d\varphi \tag{19}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4} \int_{\varphi_1}^{\varphi_2} \frac{1}{2} + \frac{\cos(2\varphi)}{2} d\varphi \tag{20}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{w_0^4} \left[\frac{\varphi}{2} + \frac{\sin(2\varphi)}{4} \right]_{\varphi_1}^{\varphi_2} \tag{21}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{2w_0^4} \left(\varphi_2 - \varphi_1 + \frac{1}{2} \left(\sin(2\varphi_2) - \sin(2\varphi_1) \right) \right) \tag{22}$$

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{2w_0^4} \left(\varphi_2 - \varphi_1 + \cos(\varphi_2 + \varphi_1) \sin(\varphi_2 - \varphi_1) \right) \tag{23}$$

If we define $\varphi \equiv \varphi_2 \equiv -\varphi_1$, then

$$\int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{z_R}{2w_0^4} \left(2\varphi + \sin(2\varphi) \right) \tag{24}$$

For $\varphi \in [0, \pi/2]$, this integral is maximized for $\varphi = \pi/2$

$$\max \int_{L_1}^{L_2} \frac{dz}{w(z)^4} = \frac{\pi z_R}{2w_0^4} \tag{25}$$