Updated Substrate Thermorefractive Noise

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1. INTRODUCTION

The previous formula for Substrate Thermorefractive Noise was

$$S_L^{\mathrm{STR}}(\Omega) = \frac{4k_{\mathrm{B}}\kappa_{\mathrm{s}}T^2\beta_{\mathrm{s}}^2 h}{\pi(C_{\mathrm{s}}\rho_{\mathrm{s}}w^2\Omega)^2}. \tag{1}$$

However, we missed some factors from Benthem and Levin 2009

$$S_L^{\mathrm{STR,new}}(\Omega) = \frac{4k_{\mathrm{B}}\kappa_{\mathrm{s}}T^2\beta_{\mathrm{s}}^2}{\pi(C_{\mathrm{s}}\rho_{\mathrm{s}}w^2\Omega)^2}\frac{h}{\cos(\theta_2)}\frac{\eta+\eta^{-1}}{2\eta^2}\left[1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)}\right] \tag{2}$$

θ_2 is the angle of refraction in the beamsplitter, η is the ratio of the major and minor axes of the beam's elliptical cross section, l_{th} is the thermal diffusion length. Note that because we are in the beam splitter, $k = 2\pi n/\lambda \approx 1/0.07\ \mu m$. The rest of the terms are defined in our paper.

$$\eta = \frac{\cos(\theta_2)}{\cos(\theta_1)}, \tag{3}$$

where θ_1 is the angle of incidence on the beam splitter, $\theta_1 = 45°$ nominally. This formula isn't in the Benthem and Levin paper but agrees with their results : for $\theta_1 = 45°$, $\eta = 1.23$ for fused silica as they claim. For $\theta_1 = 45°$, $\eta = 1.38$ for silicon.

For completeness,

$$\theta_2 = \sin^{-1}\left(\frac{1}{n}\sin(\theta_1)\right), \tag{4}$$

l_{th} is the thermal diffusion length,

$$l_{\mathrm{th}} = \sqrt{\frac{\kappa}{C\rho\Omega}} \approx 1.44\ \mu m \text{ at } 17.6\text{ MHz}. \tag{5}$$

$kl_{\mathrm{th}} \approx 40$ in our measurement band:

$$1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)} \approx 1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(2kl_{\mathrm{th}})^4} \tag{6}$$

$$1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)} \approx 1+\frac{w^2\eta}{(\eta+\eta^{-1})8k^2l_{\mathrm{th}}^4} \tag{7}$$

$$1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)} \approx 1+\frac{w^2}{8k^2l_{\mathrm{th}}^4} \tag{8}$$

$$1+\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)} \approx \frac{w^2}{8k^2l_{\mathrm{th}}^4} \approx 1300 \tag{9}$$

So the amplitude is 36 times larger than we had thought.

2. MORE ON THE WHITE NOISE FREQUENCY BAND

Looking at fig. 1 of the Benthem and Levin paper, the ASD is flat for a range of frequencies where the standing wave contribution (the last term) dominates. Below is the frequency range and the PSD value:

Define the standing wave contribution as "dominating" when

$$\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_{\mathrm{th}})^4)} > 1 \tag{10}$$

Churning through algebra, we get

$$\Omega \gtrsim \frac{4k\kappa}{C\rho w}\sqrt{\frac{\eta+\eta^{-1}}{2\eta}} \approx \frac{4k\kappa}{C\rho w} \tag{11}$$

$$f_{\text{min, GQuEST}} \gtrsim 600\text{ kHz} \tag{12}$$

$$f_{\text{min, Geo}} \gtrsim 600\text{ Hz} \tag{13}$$

This latter value agrees with the Benthem and Levin text.

For the higher frequencies, Benthem and Levin write "thermal diffusion length becomes comparable to the wavelength of the beam light, and the 1/f dependence is recovered but at much higher value than would be predicted by the BV formula."

This doesn't align with their value of 39 MHz for Geo600. Instead, the maximum frequency of this white noise frequency range is when

$$l_{\mathrm{th}} \gtrsim \frac{1}{2}\frac{1}{k} = \frac{\lambda}{4\pi n} \tag{14}$$

This latter value is the 1 over the wavenumber of the standing wave in the beam splitter.

$$\Omega \lesssim \frac{16\pi^2 n^2 \kappa}{C\rho\lambda^2} \tag{15}$$

$$f_{\text{max, GQuEST}} \lesssim 30 \text{ GHz} \tag{16}$$

Yes, GHz, so we're not going to geet this rolloff.

$$f_{\text{max, Geo}} \lesssim 39 \text{ MHz} \tag{17}$$

This latter value also agrees with the Benthem and Levin text.

Finally, the PSD value:

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{4k_\text{B}\kappa_\text{s}T^2\beta_\text{s}^2}{\pi(C_\text{s}\rho_\text{s}w^2\Omega)^2}\frac{h}{\cos(\theta_2)}\frac{\eta+\eta^{-1}}{2\eta^2}\frac{2k^2w^2\eta}{(\eta+\eta^{-1})(1+(2kl_\text{th})^4)} \tag{18}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{4k_\text{B}\kappa_\text{s}T^2\beta_\text{s}^2}{\pi(C_\text{s}\rho_\text{s}w^2\Omega)^2}\frac{h}{\cos(\theta_2)}\frac{k^2w^2}{\eta(2kl_\text{th})^4} \tag{19}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}\kappa_\text{s}T^2\beta_\text{s}^2}{4\pi(C_\text{s}\rho_\text{s}w^2\Omega)^2}\frac{h}{\cos(\theta_2)}\frac{w^2}{\eta k^2}\frac{C^2\rho^2\Omega^2}{\kappa^2} \tag{20}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}T^2\beta_\text{s}^2}{4\pi w^2k^2\eta\kappa}\frac{h}{\cos(\theta_2)} \tag{21}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}T^2\beta_\text{s}^2\lambda^2}{16\pi^3n^2w^2\eta\kappa}\frac{h}{\cos(\theta_2)} \tag{22}$$

$$S_L^{\text{STR,new}}(\Omega) \approx (6\cdot 10^{-21}\text{m}/\sqrt{\text{Hz}})^2 \text{ for GQuEST} \tag{23}$$

3. BEAM SPLITTER TRANSFER FUNCTION

We are partially saved by the Beam Splitter Transfer Function: the measured noise at the interferometer output is modulated by the transfer function for phase modulations imparted at the beamsplitter $H(\Omega) = \cos^2(\Omega L/c) \leq 1$. This antenna function originates from the phase modulation on the transmitted beam destructively interfering with the unmodulated reflected beam at the output port near the FSR of the arms. Thus, the total thermorefractive noise measured is $H(\Omega)S_L^{\text{STR,new}}(\Omega)$.

4. PAST EXPERIMENTAL OBSERVATION

I included is noise source for the holometer, taking into account the different beam splitter material, beam splitter spot size, beam splitter thickness, wavelength, and arm length from GQuEST. Looking at Fig. 12 in Chou 2017, this STR Noise model is only 2x lower than where the cross correlated ASDs bottom out. Thus the holometer may have seen this noise.

5. MITIGATION STRATEGIES

This STR Noise amplitude is pretty large. Can we change materials?

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}T^2h\lambda^2\cos(\theta_1)}{16\pi^3w^2}\frac{\beta_\text{s}^2}{\kappa n^2\cos^2(\theta_2)} \tag{24}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}T^2h\lambda^2\cos(\theta_1)}{16\pi^3w^2}\frac{\beta_\text{s}^2}{\kappa n^2(1-\sin^2(\theta_1)/n^2)} \tag{25}$$

$$S_L^{\text{STR,new}}(\Omega) \approx \frac{k_\text{B}T^2h\lambda^2\cos(\theta_1)}{16\pi^3w^2}\frac{\beta_\text{s}^2}{\kappa(n^2-\sin^2(\theta_1))} \tag{26}$$

The second fraction has all of the material property dependence.

Some values for $\beta^2/(\kappa*(n^2-\sin^2(\theta_1)))$ in m/(W K):
c-Si, 294 K: $1.5\cdot 10^{-11}$
sapphire 294 K: $1\cdot 10^{-13}$
CaF_2, 294 K: $8\cdot 10^{-12}$
fused silica, 294 K: $3\cdot 10^{-11}$

Sapphire seems like an appealing choice and the GWINC model confirms it. Sapphire has a very high birefringence (10^-2 compared to 10^-7 for silicon) which is experimentally difficult and is an unmodeled noise source.

Another alternative is a cryogenic beamsplitter. This has the added benefit of reducing thermal lensing as well:

Some values for $T^2\beta^2/(\kappa*(n^2-\sin^2(\theta_1)))$ in (m K)/W:
c-Si, 294 K: $1.4\cdot 10^{-6}$
c-Si, 123 K: $1.7\cdot 10^{-8}$
c-Si, 77 K: $6.5\cdot 10^{-10}$
sapphire 294 K: $1\cdot 10^{-8}$
CaF_2, 294 K: $8\cdot 10^{-7}$
fused silica, 294 K: $3\cdot 10^{-6}$

6. IMPLICATIONS FOR MORE NOISE SOURCES

Finally, a parting thought on including the standing wave contributions for other noise sources.

Charge Carrier noise includes the standing wave contribution due to Siegel and Levin's 2023 paper on it.

There are more noise sources, however: The optical path length of the transmitted beam in the beam splitter changes due to thermal and mechanical fluctuations. Is this a noise source to consider?

I was working on including a standing wave contribution to Coating Thermo Refractive Noise. Will this have a large effect? Maybe even some on thermo-elastic noise?